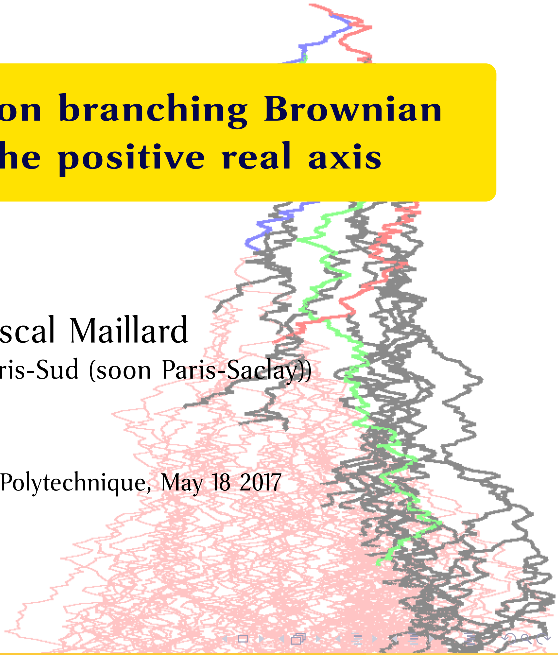
Recent results on branching Brownian motion on the positive real axis



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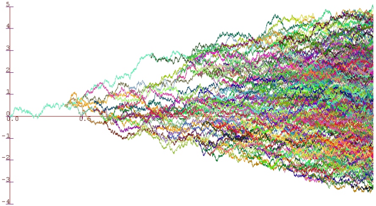
Outline

Branching Brownian motion (BBM)



Picture by Matt Roberts

Definition

- A particle performs standard Brownian motion started at a point $x \in \mathbb{R}$.
- With rate 1/2, it branches into 2 offspring (can be generalized)
- Each offspring repeats this process independently of the others.

$\longrightarrow$ A Brownian motion indexed by a tree.

Why BBM ?

- Discrete counterpart: branching random walk, has lots of applications in diverse domains
  - Generalisation of age-dependent branching processes (*Crump–Mode–Jagers process*), model for asexual population undergoing mutation (position = fitness)
  - Toy model for *log-correlated field*, e.g. 2-dimensional Gaussian free field appearing notably in *Liouville quantum gravity theory*.
  - Used to study random walk in random environment on trees Hu–Shi et al., growth-fragmentation processes Bertoin–Budd–Curien–Kortchemski, loop $O(n)$ model on random quadrangulations Chen–Curien–M., . . .
- Intimate relation with (F-)KPP equation
- With diffusion constant depending on time : also known as Derrida's CREM spin glass model

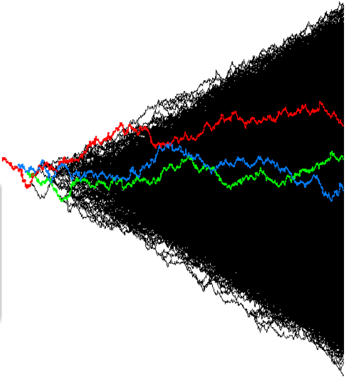
Maximum : LLN

M_t = maximum at time t.

LLN (Biggins '77)

Almost surely,

$$M_t/t \to 1, \quad \text{as } t \to \infty.$$



Picture by Éric Brunet

A family of martingales

For every $\theta \in \mathbb{R}$,

$$\mathbb{E}[\#\{u \in \mathcal{N}_t : X_u(t) \approx \theta t\}] = e^{\frac{1}{2}t}\mathbb{P}(B_t \approx \theta t) \approx e^{\frac{1}{2}(1-\theta^2)t}.$$

Martingales:

$$W_t^{(\theta)} = \sum_{u \in \mathcal{N}_t} e^{\theta X_u(t) - \frac{1}{2}(1+\theta^2)t}$$

Theorem (Biggins 78)

The martingale $(W_t^{(\theta)})_{t\geq 0}$ *is uniformly integrable if and only if* $|\theta| < 1$*. In this case, for every* $a, b \in \mathbb{R}$*,* $a < b$*,*

$$\frac{\#\{u \in \mathcal{N}_t : X_u(t) \in \theta t + [a,b]\}}{\mathbb{E}[\#\{u \in \mathcal{N}_t : X_u(t) \in \theta t + [a,b]\}]} \to W^{(\theta)} := W_\infty^{(\theta)}, \quad \text{a.s. as } t \to \infty.$$

Derivative martingale

For $\theta = 1$, $W_t^{(1)} \to 0$, almost surely as $t \to \infty$. Derivative martingale:

$$D_t = -\frac{d}{d\theta} W_t^{(\theta)} \Big|_{\theta=1} = \sum_{u \in \mathcal{N}_t} (t - X_u(t)) e^{X_u(t) - t}.$$

Theorem (Lalley-Sellke 87)

Almost surely, D_t converges as $t \to \infty$ to a non-degenerate r.v. D.

Theorem (Bramson 83 + Lalley-Sellke 87, Aïdekon 11)

Let M_t = maximum at time t. Then, conditioned on D, for some constant $C > 0$,

$$M_t - (t - \frac{3}{2} \log t) \Rightarrow \log CD + G,$$

where G is a standard Gumbel-distributed random variable.

Outline

Absorption at the origin

- Start with one particle at $x \geq 0$.
- Add *drift* $-\mu$, $\mu \in \mathbb{R}$ to motion of particles.
- Kill particles upon hitting the origin.

Theorem (Kesten 78)

$$\mathbb{P}(\textit{survival}) > 0 \iff \mu < 1.$$

Why should we do this?

- Useful for the study of BBM without absorption (e.g., convergence of derivative martingale)
- Biological interpretation: natural selection
- Appears in other mathematical models, e.g. infinite bin models Aldous, Mallein–Ramassany

Absorption at the origin, $\mu \geq 1$

Start with one particle at 0, absorb particles at $-x$. N_x = number of particles absorbed at $-x$. Set

$$\theta_\pm = \mu \pm \sqrt{\mu^2 - 1}.$$

Theorem (Neveu 87, Chauvin 88)

$(N_x)_{x\geq 0}$ is a continuous-time Galton–Watson process. Moreover, almost surely as $x \to \infty$,

- *If $\mu > 1$, $e^{-\theta_- x} N_x \to W^{(\theta_-)}$.*
- *If $\mu = 1$, $x e^{-x} N_x \to D$.*

Theorem

As $x \to \infty$,

- $\mu > 1$: $\mathbb{P}(W^{(\theta_-)} > x) \sim C(\mu) x^{-\theta_+/\theta_-}$ *Guivarc'h 90, Liu 00*
- $\mu = 1$: $\mathbb{P}(D > x) \sim 1/x$ *Buraczewski 09, Berestycki–Berestycki–Schweinsberg 10, M. 12*

Absorption at the origin, $\mu \geq 1$ (contd.)

$$\theta_\pm = \mu \pm \sqrt{\mu^2 - 1}.$$

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Theorem (M. 10, Aïdekon-Hu-Zindy 12)

As $n \to \infty$,

- $\mu > 1$: $\mathbb{P}(N_x > n) \sim C(e^{\theta_+ x} - e^{\theta_- x})/n^{-\theta_+/\theta_-}$.
- $\mu = 1$: $\mathbb{P}(N_x > n) \sim xe^x/(n(\log n)^2)$.

Outline

Absorption at the origin, $\mu = 1 - \varepsilon$

Few works on $\mu < 1$ (Berestycki-Brunet-Harris-Miloś, Corre). But near-critical case $\mu = 1 - \varepsilon$, $0 < \varepsilon \ll 1$ well understood. Parametrize ε by

$$\varepsilon = \frac{\pi^2}{2L^2} \qquad (\varepsilon \to 0 \iff L \to \infty).$$

Theorem (Brunet-Derrida 06, Gantert-Hu-Shi 08)

$$\mathbb{P}_1(\mathit{survival}) = \exp\left(-(1+o(1))L\right), \quad L \to \infty.$$

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Theorem (BBS 10)

There exists $C > 0$, such that, as $L \to \infty$,

$$\mathbb{P}_{L+x}(\textit{survival}) \to 1 - \phi(x), \quad \phi(x) := \mathbb{E}[\exp(-CDe^x)].$$

and if $x = x(L)$ such that $L - x \to \infty$,

$$\mathbb{P}_x(\textit{survival}) \sim C(L/\pi)\sin(\pi x/L)e^{x-L}.$$

BBS 10 proof

Define

$$Z_t^L = \sum_{u \in \mathcal{N}_t} L \sin(\pi X_u(t)/L) e^{x-L}.$$

Then $(Z_t^L)_{t \geq 0}$ is (almost) a martingale for BBM with absorption at 0 and at L.

Theorem (BBS 10)

Suppose the initial configurations are such that $Z_0^L \Rightarrow z_0$ as $L \to \infty$, and $L - \max_u X_u(0) \to \infty$. Then $(Z_{L^3 t}^L)_{t \geq 0}$ converges as $L \to \infty$ (wrt fidis) to a continuous-state branching process started at z_0. Moreover, $\mathbb{P}$(BBM survives forever) $\to \mathbb{P}$(CSBP started from z_0 goes to ∞).

The CSBP in the above theorem is Neveu's CSBP and has branching mechanism

$$\psi(u) = au + \pi^2 u \log u = a' u + \pi^2 \int_0^\infty (e^{-ux} - 1 + ux 1_{x \leq 1}) \frac{dx}{x^2},$$

for some (implicit) constants $a, a' \in \mathbb{R}$. In particular, it is supercritical (with ∞ mean).

BBS 10 proof (2)

Theorem (BBS 10)

If $x = x(L)$ such that $L - x \to \infty$,

$$\mathbb{P}_x(\textit{survival}) \sim \frac{CL}{\pi} \sin(\pi x/L) e^{x-L}.$$

Proof: Set $w(x) := L\sin(\pi x/L)e^{x-L}$. Start BBM with $1/w(x)$ particles at x at time 0. Then

$$\mathbb{P}(\text{survival}) \to \mathbb{P}(\text{CSBP started at 1 goes to } \infty) \in (0,1).$$

Also, by independence,

$$1 - \mathbb{P}(\text{survival}) = (1 - \mathbb{P}_x(\text{survival}))^{1/w(x)} \sim \exp\left(-\frac{\mathbb{P}_x(\text{survival})}{w(x)}\right),$$

and so

$$\mathbb{P}_x(\text{survival}) \sim Cw(x). \quad \square$$

BBS 10 proof (3)

Theorem (BBS 10)

There exists $C > 0$, such that, as $L \to \infty$,

$$\mathbb{P}_{L+x}(\textit{survival}) \to 1 - \phi(x), \quad \phi(x) = \mathbb{E}[\exp(-CDe^x)].$$

Proof: Wait a long time T (independent of L), so that $L - \max_u X_u(T) \gg 1$. Then using $L \sin(\pi x/L) \sim \pi(L - x)$ for $L - x \ll L$, we get

$$Z_T^L \approx \pi e^x D_T,$$

with $(D_t)_{t \geq 0}$ the derivative martingale of usual BBM. Let first $L \to \infty$ then $T \to \infty$ to get

$$\begin{aligned} \mathbb{P}_{L+x}(\text{survival}) &= 1 - \mathbb{E}[\mathbb{P}_{L+x}(\text{extinction} \,|\, \mathcal{F}_T)] \\ &\approx 1 - \mathbb{E}[\mathbb{P}(\text{CSBP started from } \pi e^x D_T \text{ goes to } 0)] \\ &\approx 1 - \mathbb{E}[\exp(-CDe^x)] = 1 - \phi(x). \quad \square \end{aligned}$$

BBS 10 convergence to CSBP

Basic idea

Decompose process into bulk + fluctuations by putting an additional absorbing barrier at L.

- bulk: Particles that don't hit L.
- fluctuations: Particles from the moment they hit L.

Then,

- $Z_t^{L,\text{bulk}}$ stays bounded over time scale L^3.
- $Z_t^{L,\text{fluctuations}}$ increases from the contributions of the particles hitting L, an increase being roughly distributed as πD, with D derivative martingale limit.
- Particles hit L with rate $O(L^{-3})$.

BBS 10 convergence to CSBP

Basic idea

Decompose process into bulk + fluctuations by putting an additional absorbing barrier at $L - A$, where A is a large constant.

- bulk: Particles that don't hit $L - A$.
- fluctuations: Particles from the moment they hit $L - A$.

Then,

- $Z_t^{L,\text{bulk}}$ decreases almost deterministically as $\exp(-At/L^3)$.
- $Z_t^{L,\text{fluctuations}}$ increases from the contributions of the particles hitting L, an increase being roughly distributed as $\pi e^{-A} D$, with D derivative martingale limit.
- Particles hit $L - A$ with rate $O(e^A/L^3)$.

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- Particles hit $L - A$ with rate $O(e^A/L^3)$.

Recall: $\mathbb{P}(D > x) \sim 1/x$, $x \to \infty$. This yields convergence of $(Z^L_{L^3 t})_{t \geq 0}$ to Neveu's CSBP as $L \to \infty$.

Remarks

1. The basic phenomenological picture of BBM with near-critical drift (bulk + fluctuations) was established in Brunet–Derrida–Mueller–Munier 06
2. The techniques in BBS 10 were a key ingredient in the study of BBM with selection of the N right-most particles, $N \gg 1$ (M 16). Relation between parameters: $\log N \approx L$, so $\varepsilon \approx \pi^2/2(\log N)^2$.

Outline

Critical drift $\mu = 1$. Questions

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- Asymptotic of $\mathbb{P}_x$(survival until time t)?
- Conditioned on survival until time t, what does the BBM look like?

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Kesten 78:

- Let $L_t = ct^{1/3}$, $c = (3\pi^2/2)^{1/3}$, Fix $x \geq 0$.

$$\mathbb{P}_x(\text{survival until time } t) = xe^{x - L_t + O((\log t)^2)}.$$

- Conditioned on survival until time t, with high probability,

$$\#\mathcal{N}_t \leq e^{O(t^{2/9}(\log t)^{2/3})} \quad \text{and} \quad \max_u X_u(t) \leq O(t^{2/9}(\log t)^{2/3}).$$

Note: $t^{1/3}$ scaling reminiscent of results about particles in BBM staying always close to the maximum Faraud-Hu-Shi, Fang-Zeitouni, Roberts.

BBS 12 results

$L_t = ct^{1/3}$, $c = (3\pi^2/2)^{1/3}$, $w_t(x) = L_t \sin(\pi x/L_t) e^{x-L_t}$.

Theorem (BBS 12)

$$C_1 \le \mathbb{P}_{L_t}(\textit{survival until time } t) \le C_2.$$

If $L_t - x \ge 1$,

$$C_1 w_t(x) \le \mathbb{P}_x(\textit{survival until time } t) \le C_2 w_t(x).$$

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Theorem (Berestycki-M.-Schweinsberg, in preparation)

There exists $C > 0$, *such that, as* $t \to \infty$,

$$\mathbb{P}_{L_t+x}(\textit{survival until time } t) \to 1 - \phi(x), \quad \phi(x) = \mathbb{E}[\exp(-CDe^x)].$$

and if $x = x(t)$ *such that* $L_t - x \to \infty$,

$$\mathbb{P}_x(\textit{survival until time } t) \sim (C/\pi) w_t(x)$$

New results

$L_t = ct^{1/3}$, $c = (3\pi^2/2)^{1/3}$, ζ = time of extinction.

Corollary (BMS)

1. *For fixed $x \in \mathbb{R}$, under $\mathbb{P}_{L_t+x}$, the r.v. $(\zeta - t)/t^{2/3}$ converges in law to $\frac{3}{c}(G - x - \log CD)$, where G is a Gumbel-distributed random variable independent of D.*
2. *Suppose $L_t - x \to \infty$. Conditionally on $\zeta > t$, under $\mathbb{P}_x$, $(\zeta - t)/t^{2/3}$ converges in law to $\mathrm{Exp}(c/3)$ as $t \to \infty$.*

Reason: For fixed $s \geq 0$,

$$L_{t+st^{2/3}} = L_t + \frac{c}{3}s + o(1).$$

This gives as $t \to \infty$, for fixed $x \in \mathbb{R}$,

$$\mathbb{P}_{L_t+x}(\zeta \leq t + st^{2/3}) \to \phi(x - \frac{c}{3}s) = \mathbb{E}[e^{-CDe^{x-(c/3)s}}].$$

New results (contd.)

$L_t = ct^{1/3}$, $c = (3\pi^2/2)^{1/3}$, ζ = time of extinction, $M_t = \max_u X_u(t)$.

Theorem (BMS)

1. *For fixed $x \in \mathbb{R}$, under $\mathbb{P}_{L_t+x}$, the r.v. $M_t/t^{2/9}$ converges in law to $(3c^2(G - x - \log CD) \vee 0)^{1/3}$, where G is a Gumbel-distributed random variable independent of D.*
2. *Suppose $L_t - x \to \infty$. Conditionally on $\zeta > t$, under $\mathbb{P}_x$, $M_t/t^{2/9}$ converges in law to $(3c^2 V)^{1/3}$, where $V \sim \mathrm{Exp}(1)$.*

Reason: morally, $M_t \approx L_{\zeta - t}$ if $\zeta > t$ (and $M_t = 0$ if $\zeta \leq t$).

New results (contd.)

$L_t = ct^{1/3}$, $c = (3\pi^2/2)^{1/3}$, ζ = time of extinction, $M_t = \max_u X_u(t)$.

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Reason: morally, $M_t \approx L_{\zeta-t}$ if $\zeta > t$ (and $M_t = 0$ if $\zeta \leq t$).

Same result holds with M_t replaced by $\log \#\mathcal{N}_t$.

New results (contd.)

- $L_t(s) = L_{t-s} = c(t-s)^{1/3}$.
- $Z_t(s) = \sum_{u\in\mathbb{N}_s} w_{t-s}(X_u(s)) = \sum_{u\in\mathbb{N}_s} L_t(s)\sin(\pi X_u(s)/L_t(s))e^{X_u(s)-L_t(s)}$.

Theorem (BMS)

Suppose the initial configurations are such that $Z_t(0) \Rightarrow z_0$ as $t\to\infty$, and $L_t - \max_u X_u(0) \to \infty$. Then

- *$(Z_t(t(1-e^{-s})))_{s\geq 0}$ converges as $t\to\infty$ (wrt fidis) to the CSBP with branching mechanism $\psi(u) = au + \frac{2}{3}u\log u$ started at z_0.*
- *$\mathbb{P}(\zeta > t) \to \mathbb{P}($CSBP started from z_0 goes to $\infty)$, as $t\to\infty$.*
- *Conditioned on $\zeta > t$, $(Z_t(t(1-e^{-s})))_{s\geq 0}$ converges as $t\to\infty$ to the CSBP started at z_0 conditioned to go to ∞.*

New results (contd.)

- $L_t(s) = L_{t-s} = c(t-s)^{1/3}$.
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- $\mathbb{P}(\zeta > t) \to \mathbb{P}($*CSBP started from z_0 goes to ∞*$)$*, as $t\to\infty$.*
- *Conditioned on $\zeta > t$, $(Z_t(t(1-e^{-s})))_{s\geq 0}$ converges as $t\to\infty$ to the CSBP started at z_0 conditioned to go to ∞.*

Proof inspired by BBS 10 but requiring furthermore precise estimates for density of Brownian motion in curved domains refining those obtained in Roberts 12.

Relation between results

In order to understand the relation between the several results, we use the long-time behavior of Neveu's CSBP. It grows doubly-exponentially:

Theorem (Neveu 92)

Let $(Y_t)_{t\geq 0}$ be the CSBP with branching mechanism $\psi(u) = au + bu\log u$, $a \in \mathbb{R}$, $b > 0$, starting at $z_0 > 0$. Then,

$$\frac{\log Y_t}{e^{bt}} \text{ converges almost surely to a limit } Y.$$

In particular, almost surely, the process survives iff $Y > 0$. Furthermore, there is $C = C(a, b)$, such that $Y - \log Cz_0$ follows the Gumbel distribution.

Relation between results (contd.)

Heuristic: As long as $R_s \approx L_t(s)$, we expect $\log Z_t(s) \approx R_s - L_t(s)$. When does R_s become significantly different from $L_t(s)$?

Relation between results (contd.)

Heuristic: As long as $R_s \approx L_t(s)$, we expect $\log Z_t(s) \approx R_s - L_t(s)$. When does R_s become significantly different from $L_t(s)$?

Answer: With the asymptotic growth of Neveu's CSBP, can check that $\log Z_t(s) \ll L_t(s)$ as long as $t - s \gg t^{2/3}$, hence the turning point is at $s = t - Kt^{2/3}$ for K large and one can read off M_t as well as $(\zeta - t)/t^{2/3}$ from $Z_t(s)$ at that point.

Conclusion

1. We were able to push the techniques from BBS 10 on BBM with near-critical drift to the case of critical drift.
2. Results might be of help for the fine study of other models involving extremal particles of BBM.
   Example CREM (Derrida's continuous random energy model): BBM during time $[0, T]$ with time-dependent diffusion constant $2\sigma^2(t/T)$. If σ^2 is strictly decreasing, then (M., Zeitouni 16) there exists a function $m(T)$ and constants $c, c', c'' > 0$, such that

$$\{\text{maximum at time } T\} - m(T) \Rightarrow \text{mixture of Gumbel,}$$
$$\text{with } m(T) = cT - c' T^{2/3} - c'' \log T + O(1).$$

   Removing the $O(1)$ term would require an analysis similar to the one performed here.

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Thank you for your attention!